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1. For electric and magnetic fields, show that $B^2 - E^2$ and $\mathbf{E} \cdot \mathbf{B}$ are invariant under Lorentz transformation, i.e. they are Lorentz scalar.

   Solution. Assume WLOG that boost is in x-direction.

$$\begin{aligned}
B'^2 - E'^2 &= \mathbf{B}' \cdot \mathbf{B}' - \mathbf{E}' \cdot \mathbf{E}' \\
&= B_x^2 + \gamma^2(B_y + \beta E_z)^2 + \gamma^2(B_z - \beta E_y)^2 - (E_x^2 + \gamma^2(E_y - \beta B_z)^2 + \gamma^2(E_z + \beta B_y) \\
&= (B_x^2 + \gamma^2(1-\beta^2)B_y^2 + \gamma^2(1-\beta^2)B_z^2) - (E_x^2 + \gamma^2(1-\beta^2)E_y^2 + \gamma^2(1-\beta^2)E_z^2) \\
&= (B_x^2 + B_y^2 + B_z^2) - (E_x^2 + E_y^2 + E_z^2) \\
&= \mathbf{B} \cdot \mathbf{B} - \mathbf{E} \cdot \mathbf{E} \\
&= B^2 - E^2
\end{aligned}$$

   where

$$\begin{aligned}
\beta &= \frac{v}{c} \\
\gamma &= \frac{1}{\sqrt{1-\beta^2}}.
\end{aligned}$$

   Therefore, $B^2 - E^2$ is invariant under Lorentz transformation.

   Again, assume WLOG that boost is in x-direction and use the definitions for β and γ given above.

$$\begin{aligned}
\mathbf{E}' \cdot \mathbf{B}' &= E_x B_x + \gamma^2[(E_y - \beta B_z)(B_y + \beta E_z) + (E_z + \beta B_y)(B_z - \beta E_z)] \\
&= E_x B_x + \gamma^2[E_y B_y + \beta E_y E_z - \beta B_y B_z - \beta^2 E_z B_z \\
&+ E_z B_z - \beta E_y E_z + \beta B_y B_z - \beta^2 E_y B_y] \\
&= E_x B_x + E_y B_y + E_z B_z \\
&= \mathbf{E} \cdot \mathbf{B}.
\end{aligned}$$

   So, $\mathbf{E} \cdot \mathbf{B}$ is invariant wrt to Lorentz transformation.

2. Show that the stress tensor ${T_\mu}^\nu$ E/M fields has zero trace.

   Solution. Consider definition of elements along diagonal of ${T_\mu}^\nu$:

$$\begin{aligned}
{T_0}^0 &= \frac{1}{8\pi}(E^2 + B^2) \\
{T_i}^i &= -\frac{1}{4\pi}\left[E_i^2 + B_i^2 - \frac{1}{2}(E^2 + B^2)\right].
\end{aligned}$$

Taking the sum of the elements gives

$$\begin{aligned}
\mathrm{tr}(T_\mu{}^\nu) &= T_0{}^0 + T_1{}^1 + T_2{}^2 + T_3{}^3 \\
&= \frac{1}{8\pi}(E^2+B^2) - \sum_{i=1}^{3}\frac{1}{4\pi}\left[E_i^2 + B_i^2 - \frac{1}{2}(E^2+B^2)\right] \\
&= \frac{1}{8\pi}(E^2+B^2) - \frac{1}{4\pi}(E_1^2+E_2^2+E_3^2+B_1^2+B_2^2+B_3^2) + \frac{1}{8\pi}(E^2+B^2) \\
&= \frac{1}{4\pi}(E^2+B^2) - \frac{1}{4\pi}(E^2+B^2) \\
&= 0.
\end{aligned}$$

Therefore the stress tensor has zero trace.

3. Show that the stress tensor $T_\mu{}^\nu$ of E/M fields is divergenceless ($T^{\nu\mu}{}_\nu = 0$) in the absence of source of charge sources.

Solution. Start with the Maxwell's equations in tensor form,

$$\begin{aligned}
4\pi T^{\mu\nu}{}_{,\nu} &= F^{\mu\alpha}{}_{,\nu}F^\nu{}_\alpha + F^{\mu\alpha}F^\nu{}_{\alpha,\nu} - \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu} \\
&= F^{\mu\alpha}F^\nu{}_{\alpha,\nu} + F_{\nu\alpha}(F^{\mu\alpha,\nu} - \frac{1}{2}F^{\nu\alpha,\mu}) \\
&= F^{\mu\alpha}F^\nu{}_{\alpha,\nu} + \frac{1}{2}F_{\nu\alpha}(F^{\alpha\nu,\mu} + F^{\mu\alpha,\nu} + F^{\nu\mu,\alpha}) \\
&= 0
\end{aligned}$$

since

$$\begin{aligned}
F^\nu{}_{\alpha,\nu} = -4\pi j_\alpha &= 0 \\
F^{\alpha\nu,\mu} + F^{\mu\alpha,\nu} + F^{\nu\mu,\alpha} &= 0.
\end{aligned}$$

4. In a frame O, spatially uniform, time-independent electric and magnetic fields are in the $\mathbf{x}$ and $\mathbf{z}$ directions respectively. $\mathbf{E} = E_0\hat{\mathbf{x}}$ and $\mathbf{B} = B_0\hat{\mathbf{z}}$. Find the conditions necessary for the existence of a Lorentz frame O' in which (1) **E**' $= 0$ and (2) **B**' $= 0$.

Solution. (1) Need to make $E'_x = 0$.

$$\begin{aligned}
E'_x &= -F'^{01} = -\Lambda^0{}_\gamma\Lambda^1{}_\gamma F^{\gamma\delta} \\
&= -\Lambda^0{}_\gamma F^{\gamma 1} = -\Lambda^0{}_1 F^{01} - \Lambda^0{}_2 F^{21} \\
&= -\gamma E_0 + \beta\gamma B_0.
\end{aligned}$$

In order for $E'_x = 0$, need

$$\mathbf{v} = \frac{E_0}{B_0}c\hat{\mathbf{y}}.$$

(2) Follow the same steps for $B'_z = 0$ gives

$$\begin{aligned}
B'_z &= -F'^{21} = -\Lambda^2{}_\gamma \Lambda^1{}_\gamma F^{\gamma\delta} \\
&= -\Lambda^2{}_\gamma F^{\gamma 1} = \Lambda^2{}_0 F^{01} + \Lambda^2{}_2 F^{21} \\
&= \beta\gamma E_0 + \gamma B_0 \\
\implies \mathbf{v} &= -\frac{B_0}{E_0} c\hat{\mathbf{y}}.
\end{aligned}$$

Equating the step above with the result of (1) gives

$$E_0^2 - B_0^2 = 0.$$

This comes as no surprise, given exercise 1. This is the condition for existence of a frame O' in which (1) **E**' = 0 and (2) **B**' = 0.

5. If a Lorentz transform is

$$\Lambda^\alpha{}_\beta = \begin{bmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ \frac{v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

find the transformation of **E** and **B**.

Solution. Rewrite the tensor as

$$\Lambda^\alpha{}_\beta = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transform the fields to get

$$\begin{aligned}
E'_x &= -\Lambda^0{}_0 \Lambda^1{}_1 F^{01} - \Lambda^1{}_0 \Lambda^0{}_1 F^{10} = \gamma^2(E_x - \beta^2 E_x) = E_x \\
E'_y &= -\Lambda^0{}_\gamma F^{\gamma 2} = -\Lambda^0{}_0 F^{02} - \Lambda^0{}_1 F^{12} = \gamma E_y + \beta\gamma B_z = \gamma(E_y + \beta B_z) \\
E'_z &= -\Lambda^0{}_\gamma F^{\gamma 3} = -\Lambda^0{}_0 F^{03} - \Lambda^0{}_1 F^{13} = \gamma E_z - \beta\gamma B_z = \gamma(E_z - \beta B_y) \\
B'_x &= -\Lambda^3{}_3 F^{23} = B_x \\
B'_y &= -\Lambda^1{}_\gamma F^{23} = -\Lambda^1{}_0 F^{03} - \Lambda^1{}_1 F^{13} = \gamma(B_y - \beta E_z) \\
B'_z &= -\Lambda^1{}_\gamma F^{\gamma 2} = -\Lambda^1{}_0 F^{02} - \Lambda^1{}_1 F^{12} = \gamma(B_z + \beta E_y).
\end{aligned}$$

6. For a rest observer A, a magnetic field **B** is from a current I in an infinitely long straight wire of radius r_0, in which the charge density is zero. The second observer B travels parallel to the wire with velocity v. Find the electromagnetic fields **E**' and **B**' seen by B.

Solution. For observer A,

$$\begin{aligned}\vec{E} &= 0 \\ \vec{B}(r) &= \begin{cases} \frac{2rI}{cr_0^2}\hat{e}_\phi & r > r_0 \\ \frac{2I}{cr}\hat{e}_\phi & r < r_0 \end{cases}.\end{aligned}$$

For observer B,

$$\begin{aligned}E'_\parallel &= E_\parallel = 0 \\ B'_\parallel &= B_\parallel = 0\end{aligned}$$

in the direction of the wire. The perpendicular components are

$$\begin{aligned}\vec{E}'_\perp &= \frac{1}{\sqrt{1-v^2/c^2}}(\vec{E}_\perp + \vec{v}\times\vec{B}_\perp) \\ &= -\frac{1}{\sqrt{1-v^2/c^2}}vB\hat{e}_r \\ &= \begin{cases} -\frac{2r}{cr_0^2}\frac{vI}{\sqrt{1-v^2/c^2}}\hat{e}_r & r < r_0 \\ -\frac{2}{cr}\frac{vI}{\sqrt{1-v^2/c^2}}\hat{e}_r & r > r_0 \end{cases} \\ \vec{B}'_\perp &= \frac{1}{\sqrt{1-v^2/c^2}}\left(\vec{B}_\perp - \frac{\vec{\times}\vec{E}_\perp}{c^2}\right) \\ &= \frac{1}{\sqrt{1-v^2/c^2}}B\hat{e}_\phi \\ &= \begin{cases} \frac{2r}{cr_0^2}\frac{vI}{\sqrt{1-v^2/c^2}}\hat{e}_\phi & r < r_0 \\ \frac{2}{cr}\frac{vI}{\sqrt{1-v^2/c^2}}\hat{e}_\phi & r > r_0 \end{cases}.\end{aligned}$$